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## Velocity Representation of Free-Surface Flows and Fourier-Kochin Representation of Waves

by

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## Introduction

This study considers the basic problem of determining the flow, inside a free-surface potential-flow region, that corresponds to a given flow at the surface bounding the potential-flow region. A new fundamental boundary integral representation is given. This representation of linear free-surface potential flows, called velocity representation, explicitly defines the velocity  $\vec{u}$  within the potential-flow region in terms of the velocity  $\vec{u}$  at the boundary surface. Specifically, the velocity representation defines the velocity inside a potential-flow domain in terms of source and vortex distributions with strength equal to the normal component  $\vec{u} \cdot \vec{n}$  and the tangential component  $\vec{u} \times \vec{n}$  of the velocity at the boundary surface.

Thus, the velocity representation does not involve the velocity potential  $\phi$ . This property is a major difference between the velocity representation and the classical boundary integral representation, called potential representation hereafter, which defines the velocity potential  $\phi$  within a potential-flow region in terms of the potential  $\phi$  and its normal derivative  $\partial\phi/\partial n$  at the boundary surface. The velocity representation can therefore be used to couple a viscous flow, for which a velocity potential cannot be defined, and a potential flow in a direct manner, i.e. without having to solve an integral equation as is necessary if the potential representation is used. Specifically, the potential representation requires a two-step procedure: an integral equation must first be solved to determine  $\phi$  at the boundary surface from the known boundary value of  $\partial\phi/\partial n$ , and  $\phi$  and  $\nabla\phi$  can subsequently be determined at interior points from the boundary values of  $\phi$  and  $\partial\phi/\partial n$ .

Another important difference between the classical potential representation and the velocity representation given here is that the velocity representation defines  $\vec{u}$  in terms of first derivatives of the Green function  $G$ , whereas  $\vec{u}$  can only be obtained from the potential representation using either analytical differentiation of the potential representation, which involves second-order derivatives of  $G$ , or numerical differentiation of  $\phi$ .

The new velocity representation and, for completeness, the classical potential representation are given here for generic free-surface potential flows and for four basic classes of flows corresponding to the particular cases of (i) an infinitely rigid or soft free-surface plane, (ii) diffraction-radiation of time-harmonic water waves without forward speed, (iii) steady ship waves, and (iv) time-harmonic ship waves (diffraction-radiation with forward speed). These flow representations, given here for deep water, can be extended to uniform finite water depth, and indeed can be extended to a broader class of problems involving dispersive waves that propagate in the presence of a planar boundary.

The velocity representation provides a new mathematical basis for computing free-surface potential flows about nonlifting and lifting bodies using free-surface Green functions. In particular, this boundary-integral representation can be used together with the Fourier-Kochin approach expounded in Noblesse and Yang [1], as is indeed shown here. Specifically, the velocity representation and the Fourier-Kochin approach are shown to yield remarkably simple analytical representations of the waves generated by an arbitrary boundary velocity distribution for time-harmonic flows, with and without forward speed, and for steady flows. The importance of the Fourier-Kochin representations of waves for practical purposes is demonstrated, for the case of steady ship waves, in Guillerm and Alessandrini [2] and Yang et al. [3,4]. Specifically, the Fourier-Kochin representation of steady ship waves is coupled with nearfield calculations based on the RANS and the Euler equations in Guillerm and Alessandrini [2] and Yang et al. [3], respectively, and is applied to the design of a wave cancellation multihull ship in Yang et al. [4].

Coordinates  $(\xi, \eta, \zeta)$  and  $(x, y, z)$  are nondimensional with respect to a reference length  $L$  that characterizes the size of the ship or offshore structure. The velocity  $\vec{u} = (u, v, w)$  is nondimensional with respect to a characteristic reference velocity  $U$ , e.g. the forward speed  $\mathcal{U}$  of the ship, and the velocity potential  $\phi$  is nondimensional with respect to the reference potential  $UL$ . The  $z$  axis is vertical and points upward, and the mean free-surface plane is taken as the plane  $z=0$ . For steady and time-harmonic flow about a ship advancing with speed  $\mathcal{U}$  in calm water or in waves, the  $x$  axis is chosen along the path of the ship and points toward the ship bow, and the velocity  $\vec{u}$  corresponds to the flow disturbance due to the ship.

Consider a potential-flow domain bounded by a closed boundary surface  $\Sigma$  defined as

$$\Sigma = \Sigma^{HW} \cup \Sigma^F \cup \Sigma^\infty \quad \text{with} \quad \Sigma^{HW} = \Sigma^H \cup \Sigma^W$$

The surface  $\Sigma^H$  is an arbitrary control surface outside the viscous boundary layer that surrounds a ship hull (or other body, e.g. an offshore structure) at or below the free surface. If viscous effects are ignored,  $\Sigma^H$  may be taken as the wetted surface of the ship hull. The surface  $\Sigma^W$  represents the outer edge of the viscous wake trailing the ship, or a control surface outside the viscous wake. For a ship equipped with lifting surfaces, e.g. a sailboat,  $\Sigma^W$  includes the two sides of every vortex sheet behind the ship hull. For a multihull ship, the hull+wake surface  $\Sigma^{HW}$  consists of several component surfaces, which correspond to the separate hull components of the ship and their wakes. The surface  $\Sigma^{HW}$  is bounded upward by the mean free-surface plane  $z=0$ . The surface  $\Sigma^F$  stands for the portion of the mean free-surface plane outside  $\Sigma^{HW}$ . The surface  $\Sigma^\infty$  represents a large

boundary surface that surrounds  $\Sigma^H$ .  $\Gamma^{HW}$  and  $\Gamma^\infty$  are the intersection curves between the free surface  $\Sigma^F$  and the surfaces  $\Sigma^{HW}$  and  $\Sigma^\infty$ .

Boundary integral representations are obtained in the study, for both the potential  $\phi$  and the velocity  $\vec{u} = \nabla\phi$ , using elementary identities in vector calculus (divergence theorem and Stokes' theorem) in the 3D flow region bounded by the closed surface  $\Sigma$  and the 2D free-surface region  $\Sigma^F$  bounded by the curve  $\Gamma^{HW} \cup \Gamma^\infty$ . The surface  $\Sigma^\infty$  and the curve  $\Gamma^\infty$  are not explicitly mentioned hereafter because they yield contributions that vanish in the limit when the surface  $\Sigma^\infty$  is chosen infinitely far from the surface  $\Sigma^H$ .

The unit vector  $\vec{n} = (n^x, n^y, n^z)$  normal to the boundary surface  $\Sigma$  points inside the mean flow domain. Thus, the unit vector normal to the mean free surface  $\Sigma^F$  is  $\vec{n} = (0, 0, -1)$ . The unit vector  $\vec{t} = (t^x, t^y, 0)$  tangent to the boundary curve  $\Gamma^{HW} \cup \Gamma^\infty$  is oriented clockwise (looking down) along  $\Gamma^{HW}$  and counterclockwise along  $\Gamma^\infty$ . The unit vector normal to the boundary curve  $\Gamma^{HW}$  points inside the mean flow domain, like  $\vec{n}$ , and is given by  $\vec{\nu} = (-t^y, t^x, 0)$ .

### Potential and velocity representations for elementary Rankine singularities

The velocity field  $\vec{u} = (u, v, w)$  at a point  $\vec{\xi} = (\xi, \eta, \zeta)$  inside the flow domain is given by  $\vec{u} = \nabla_\xi \phi(\vec{\xi})$  where  $\nabla_\xi = (\partial_\xi, \partial_\eta, \partial_\zeta)$ . The potential  $\phi$  can be expressed as

$$\phi = \psi - \chi \quad (1a)$$

The potentials  $\psi$  and  $\chi$  respectively correspond to distributions of sources and dipoles over the boundary surface  $\Sigma$ :

$$\psi(\vec{\xi}) = \int_\Sigma d\mathcal{A}(\vec{x}) \vec{u}(\vec{x}) \cdot \vec{n}(\vec{x}) G(\vec{\xi}, \vec{x}) \quad (1b)$$

$$\chi(\vec{\xi}) = \int_\Sigma d\mathcal{A}(\vec{x}) \phi(\vec{x}) \nabla_x G(\vec{\xi}, \vec{x}) \cdot \vec{n}(\vec{x}) \quad (1c)$$

Here  $\nabla_x = (\partial_x, \partial_y, \partial_z)$  and the Green function  $G$  is the potential of a simple Rankine source in an unbounded fluid, i.e.

$$4\pi G = -1/r \quad \text{with} \quad r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2} \quad (2)$$

The velocity derived from the potential representation (1) is given by

$$\vec{u} = \nabla_\xi \phi = \nabla_\xi \psi - \nabla_\xi \chi = \vec{u}^\psi - \vec{u}^\chi \quad (3a)$$

The velocity components  $\vec{u}^\psi = \nabla_\xi \psi$  and  $\vec{u}^\chi = \nabla_\xi \chi$  are given by

$$\vec{u}^\psi(\vec{\xi}) = \int_\Sigma d\mathcal{A}(\vec{x}) \vec{u}(\vec{x}) \cdot \vec{n}(\vec{x}) \nabla_\xi G(\vec{\xi}, \vec{x}) \quad (3b)$$

$$\vec{u}^\chi(\vec{\xi}) = \int_\Sigma d\mathcal{A}(\vec{x}) \phi(\vec{x}) \nabla_\xi [\nabla_x G(\vec{\xi}, \vec{x}) \cdot \vec{n}(\vec{x})] \quad (3c)$$

The classical representation of the velocity  $\vec{u}$  given by (3) involves the potential  $\phi$  in (3c). The flow representation (3) is then called the potential representation hereafter. The source component  $\vec{u}^\psi$  is defined in terms of the first derivatives of the Green function  $G$ . However, the dipole component  $\vec{u}^\chi$  involves second derivatives of  $G$ . Another representation of the component  $\vec{u}^\chi$  that is defined in terms of a vortex sheet at the boundary surface and only involves first derivatives of  $G$  is

$$\vec{u}^\chi(\vec{\xi}) = \int_\Sigma d\mathcal{A}(\vec{x}) [\vec{u}(\vec{x}) \times \vec{n}(\vec{x})] \times \nabla_\xi G(\vec{\xi}, \vec{x}) \quad (4)$$

The alternative representation (4) of the velocity component  $\vec{u}^\chi$  is identical to (3c) if

$$\int_\Sigma d\mathcal{A} [\phi \nabla_\xi (\nabla_x G \cdot \vec{n}) + \nabla_\xi G \times (\nabla_x \phi \times \vec{n})] = 0 \quad (5)$$

The relations  $\nabla_\xi G = -\nabla_x G$  and  $\nabla_x^2 G = 0$ , which readily follow from (2), show that the  $x, y, z$  components of (5) are given by  $\vec{V}^x \cdot \vec{n}, \vec{V}^y \cdot \vec{n}, \vec{V}^z \cdot \vec{n}$  where the vectors  $\vec{V}^x, \vec{V}^y, \vec{V}^z$  are defined as

$$\vec{V}^x = \begin{Bmatrix} (\phi G_y)_y + (\phi G_z)_z \\ -(\phi G_y)_x \\ -(\phi G_z)_x \end{Bmatrix} \quad \vec{V}^y = \begin{Bmatrix} -(\phi G_x)_y \\ (\phi G_z)_z + (\phi G_x)_x \\ -(\phi G_z)_y \end{Bmatrix} \quad \vec{V}^z = \begin{Bmatrix} -(\phi G_x)_z \\ -(\phi G_y)_z \\ (\phi G_x)_x + (\phi G_y)_y \end{Bmatrix} \quad (6)$$

We have  $\nabla \cdot \vec{V}^x = 0, \nabla \cdot \vec{V}^y = 0, \nabla \cdot \vec{V}^z = 0$ . The divergence theorem applied to the vector fields  $\vec{V}^x, \vec{V}^y, \vec{V}^z$  and the closed surface  $\Sigma$  then shows that the  $x, y, z$  components of (5) are null.

By substituting (3b) and (4) into (3a) we obtain

$$\vec{u}(\vec{\xi}) = \int_\Sigma d\mathcal{A}(\vec{x}) \left( [\vec{u}(\vec{x}) \cdot \vec{n}(\vec{x})] \nabla_\xi G(\vec{\xi}, \vec{x}) - [\vec{u}(\vec{x}) \times \vec{n}(\vec{x})] \times \nabla_\xi G(\vec{\xi}, \vec{x}) \right) \quad (7a)$$

which yields

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \int_\Sigma d\mathcal{A} \begin{Bmatrix} \vec{u} \cdot \vec{n} G_\xi + (\vec{u} \times \vec{n})^z G_\eta - (\vec{u} \times \vec{n})^y G_\zeta \\ \vec{u} \cdot \vec{n} G_\eta + (\vec{u} \times \vec{n})^x G_\zeta - (\vec{u} \times \vec{n})^z G_\xi \\ \vec{u} \cdot \vec{n} G_\zeta + (\vec{u} \times \vec{n})^y G_\xi - (\vec{u} \times \vec{n})^x G_\eta \end{Bmatrix} \quad (7b)$$

This expression defines the velocity  $\vec{u}(\vec{\xi})$  at a point  $\vec{\xi}$  inside a flow domain, bounded by a boundary surface  $\Sigma$ , in terms of source and vortex distributions with strength equal to the normal and tangential velocity components  $\vec{u} \cdot \vec{n}$  and  $\vec{u} \times \vec{n}$  of the velocity  $\vec{u}(\vec{x})$  at  $\Sigma$ . The flow representation (7),

which defines  $\vec{u}(\vec{\xi})$  in terms of the boundary velocity distribution  $\vec{u}(\vec{x})$ , is called velocity representation. This representation only involves the velocity  $\vec{u}$  and first derivatives of the Green function  $G$ , whereas the potential representation (3) involves the potential  $\phi$  and second derivatives of  $G$ .

### Velocity and potential representations for generic free-surface flows

The potential representation (1) and the related representation (3) for the velocity hold for a generic Green function  $G$  of the form

$$4\pi G = -1/r + G^H$$

Here, the function  $G^H$  is harmonic, i.e. satisfies the Laplace equation  $\nabla^2 G^H = 0$ , in the region bounded by the surface  $\Sigma$ . Green functions associated with boundary conditions at the mean free-surface plane  $z = 0$  can be expressed as

$$G(\vec{\xi}, \vec{x}) = G^S(x - \xi, y - \eta, z - \zeta) + G^F(x - \xi, y - \eta, z + \zeta) \quad (8a)$$

$G^S$  corresponds to a simple Rankine source and is given by

$$4\pi G^S = -1/r \quad \text{with} \quad r = \sqrt{(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2} \quad (8b)$$

$G^F$  accounts for free-surface effects and is harmonic in the region bounded by  $\Sigma$ . The simple-singularity component  $G^S$  is a function of the three variables  $(x - \xi, y - \eta, z - \zeta)$ . The free-surface component  $G^F$  is a function of  $(x - \xi, y - \eta, z + \zeta)$  and satisfies the relations

$$(G_\xi^F, G_\eta^F, G_\zeta^F) = (-G_x^F, -G_y^F, G_z^F) \quad \text{and} \quad \nabla_x^2 G^F = 0 \quad (9)$$

The potential representation (3) and the decomposition (8a) of the Green function  $G$  show that the velocity field  $\vec{u}$  can be expressed as

$$\vec{u} = \vec{u}^S + \vec{u}^F \quad (10a)$$

The components  $\vec{u}^S$  and  $\vec{u}^F$  correspond to  $G^S$  and  $G^F$  in (8a). Expressions (8b) and (2) show that the simple-singularity component  $\vec{u}^S$  can be expressed in the form (7b), i.e.

$$\begin{pmatrix} u^S \\ v^S \\ w^S \end{pmatrix} = \int_{\Sigma} dA \begin{pmatrix} -\vec{u} \cdot \vec{n} G_x^S - (\vec{u} \times \vec{n})^z G_y^S + (\vec{u} \times \vec{n})^y G_z^S \\ -\vec{u} \cdot \vec{n} G_y^S - (\vec{u} \times \vec{n})^x G_z^S + (\vec{u} \times \vec{n})^z G_x^S \\ -\vec{u} \cdot \vec{n} G_z^S - (\vec{u} \times \vec{n})^y G_x^S + (\vec{u} \times \vec{n})^x G_y^S \end{pmatrix} \quad (10b)$$



It is shown in Appendix 1 that the free-surface component  $\vec{u}^F$  can similarly be expressed as

$$\begin{pmatrix} u^F \\ v^F \\ -w^F \end{pmatrix} = \int_{\Sigma} d\mathcal{A} \begin{pmatrix} -\vec{u} \cdot \vec{n} G_x^F - (\vec{u} \times \vec{n})^z G_y^F + (\vec{u} \times \vec{n})^y G_z^F \\ -\vec{u} \cdot \vec{n} G_y^F - (\vec{u} \times \vec{n})^x G_z^F + (\vec{u} \times \vec{n})^z G_x^F \\ -\vec{u} \cdot \vec{n} G_z^F - (\vec{u} \times \vec{n})^y G_x^F + (\vec{u} \times \vec{n})^x G_y^F \end{pmatrix} \quad (10c)$$

The representations (10b), (10c) can be expressed in compact vectorial forms similar to (7a)

$$\vec{u}^S = \int_{\Sigma} d\mathcal{A} [(\vec{u} \times \vec{n}) \times \nabla_x G^S - (\vec{u} \cdot \vec{n}) \nabla_x G^S] \quad (10d)$$

$$(u^F, v^F, -w^F) = \int_{\Sigma} d\mathcal{A} [(\vec{u} \times \vec{n}) \times \nabla_x G^F - (\vec{u} \cdot \vec{n}) \nabla_x G^F] \quad (10e)$$

Expressions (10), (1), (8a) yield

$$\begin{pmatrix} \phi \\ u \\ v \\ w \end{pmatrix} = \int_{\Sigma} d\mathcal{A} \begin{pmatrix} \vec{u} \cdot \vec{n} G^+ - \phi (n^x G_x^+ + n^y G_y^+ + n^z G_z^+) \\ -\vec{u} \cdot \vec{n} G_x^+ - (\vec{u} \times \vec{n})^z G_y^+ + (\vec{u} \times \vec{n})^y G_z^+ \\ -\vec{u} \cdot \vec{n} G_y^+ - (\vec{u} \times \vec{n})^x G_z^+ + (\vec{u} \times \vec{n})^z G_x^+ \\ -\vec{u} \cdot \vec{n} G_z^+ - (\vec{u} \times \vec{n})^y G_x^+ + (\vec{u} \times \vec{n})^x G_y^+ \end{pmatrix} \quad (11a)$$

where  $G^{\pm}$  is defined as

$$G^{\pm} = G^S \pm G^F \quad (11b)$$

The free-surface component  $G^F$  in (8a) and (11b) can be expressed in the alternative forms

$$4\pi G^F = \begin{pmatrix} 1/r' + H \\ -1/r' + \mathcal{H} \end{pmatrix} \quad (12a)$$

with

$$r' = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z+\zeta)^2} \quad (12b)$$

Expressions (11b), (8b) and (12a) yield the alternative representations

$$4\pi G^{\pm} = \begin{pmatrix} -1/r \pm 1/r' \pm H \\ -1/r \mp 1/r' \pm \mathcal{H} \end{pmatrix} \quad (13)$$

By substituting (13) into (11a) we can express  $\phi$  and  $\vec{u}$  in the alternative forms

$$\phi = \begin{pmatrix} \phi^R - \phi_*^R + \phi^H \\ \phi^R + \phi_*^R + \phi^{\mathcal{H}} \end{pmatrix} \quad \vec{u} = \begin{pmatrix} \vec{u}^R - \vec{u}_*^R + \vec{u}^H \\ \vec{u}^R + \vec{u}_*^R + \vec{u}^{\mathcal{H}} \end{pmatrix} \quad (14)$$

The components  $\phi^R$ ,  $\vec{u}^R$  and  $\phi_*^R$ ,  $\vec{u}_*^R$  are defined by distributions of elementary Rankine singularities  $1/r$  and  $1/r'$ . Specifically, the Rankine component  $\phi^R$ ,  $\vec{u}^R$  is given by

$$4\pi \begin{pmatrix} -\phi^R \\ u^R \\ v^R \\ w^R \end{pmatrix} = \int_{\Sigma} d\mathcal{A} \begin{pmatrix} -(\vec{u} \cdot \vec{n}) R + \phi (n^x R_x + n^y R_y + n^z R_z) \\ -(\vec{u} \cdot \vec{n}) R_x - (\vec{u} \times \vec{n})^z R_y + (\vec{u} \times \vec{n})^y R_z \\ -(\vec{u} \cdot \vec{n}) R_y - (\vec{u} \times \vec{n})^x R_z + (\vec{u} \times \vec{n})^z R_x \\ -(\vec{u} \cdot \vec{n}) R_z - (\vec{u} \times \vec{n})^y R_x + (\vec{u} \times \vec{n})^x R_y \end{pmatrix} \quad (15a)$$

with  $R = -1/r$ . The Rankine component  $\phi_*^R$ ,  $\vec{u}_*^R$  is given by

$$4\pi \begin{pmatrix} -\phi_*^R \\ u_*^R \\ v_*^R \\ -w_*^R \end{pmatrix} = \int_{\Sigma} d\mathcal{A} \begin{pmatrix} -(\vec{u} \cdot \vec{n}) R^* + \phi (n^x R_x^* + n^y R_y^* + n^z R_z^*) \\ -(\vec{u} \cdot \vec{n}) R_x^* - (\vec{u} \times \vec{n})^z R_y^* + (\vec{u} \times \vec{n})^y R_z^* \\ -(\vec{u} \cdot \vec{n}) R_y^* - (\vec{u} \times \vec{n})^x R_z^* + (\vec{u} \times \vec{n})^z R_x^* \\ -(\vec{u} \cdot \vec{n}) R_z^* - (\vec{u} \times \vec{n})^y R_x^* + (\vec{u} \times \vec{n})^x R_y^* \end{pmatrix} \quad (15b)$$

with  $R^* = -1/r'$ . The components  $\phi^H$ ,  $\vec{u}^H$  and  $\phi^{\mathcal{H}}$ ,  $\vec{u}^{\mathcal{H}}$  are defined in terms of the functions  $H$  and  $\mathcal{H}$ , which account for free-surface effects. The free-surface component  $\phi^{\mathcal{F}}$ ,  $\vec{u}^{\mathcal{F}}$  — where  $\mathcal{F}$  stands for either  $H$  or  $\mathcal{H}$  — is given by

$$4\pi \begin{pmatrix} -\phi^{\mathcal{F}} \\ u^{\mathcal{F}} \\ v^{\mathcal{F}} \\ -w^{\mathcal{F}} \end{pmatrix} = \int_{\Sigma} d\mathcal{A} \begin{pmatrix} -(\vec{u} \cdot \vec{n}) \mathcal{F} + \phi (n^x \mathcal{F}_x + n^y \mathcal{F}_y + n^z \mathcal{F}_z) \\ -(\vec{u} \cdot \vec{n}) \mathcal{F}_x - (\vec{u} \times \vec{n})^z \mathcal{F}_y + (\vec{u} \times \vec{n})^y \mathcal{F}_z \\ -(\vec{u} \cdot \vec{n}) \mathcal{F}_y - (\vec{u} \times \vec{n})^x \mathcal{F}_z + (\vec{u} \times \vec{n})^z \mathcal{F}_x \\ -(\vec{u} \cdot \vec{n}) \mathcal{F}_z - (\vec{u} \times \vec{n})^y \mathcal{F}_x + (\vec{u} \times \vec{n})^x \mathcal{F}_y \end{pmatrix} \quad (16)$$

The flow representation (14)-(16) is valid for a generic Green function  $G$  of the type defined by (8) and (12). This representation defines a free-surface potential flow in terms of the flow at a boundary surface  $\Sigma$ , which includes the free-surface plane  $\Sigma^F$ . The contribution of  $\Sigma^F$  is now considered.

### Free-surface contribution for generic free-surface flows

The contribution of the mean free surface  $\Sigma^F$ , where  $z = 0$  and  $(n^x, n^y, n^z) = (0, 0, -1)$ , to the potential and the velocity (11a) is given by

$$\begin{pmatrix} \phi \\ u \\ v \\ w \end{pmatrix} = \int_{\Sigma^F} d\mathcal{A} \begin{pmatrix} -w G^+ + \phi G_z^+ \\ w G_x^+ + u G_z^+ \\ w G_y^+ + v G_z^+ \\ w G_z^- - u G_x^- - v G_y^- \end{pmatrix}$$

At the free-surface plane  $z=0$ , (13) yields

$$4\pi (G^+, G_x^+, G_y^+, G_z^+) = (H, H_x, H_y, \mathcal{H}_z)$$

$$4\pi (G^-, G_x^-, G_y^-, G_z^-) = -(\mathcal{H}, \mathcal{H}_x, \mathcal{H}_y, H_z)$$

These relations show that the contribution of the mean free surface  $\Sigma^F$  is given by

$$4\pi \begin{pmatrix} \phi \\ u \\ v \\ w \end{pmatrix} = \int_{\Sigma^F} d\mathcal{A} \begin{pmatrix} -w H + \phi \mathcal{H}_z \\ w H_x + u \mathcal{H}_z \\ w H_y + v \mathcal{H}_z \\ u \mathcal{H}_x + v \mathcal{H}_y - w H_z \end{pmatrix} \quad (17)$$

The free-surface contribution (17) is considered below. The two simple special cases of an infinitely rigid or infinitely soft free surface are examined first. Three basic classes of free-surface flows, corresponding to diffraction-radiation of time-harmonic water waves without forward speed, steady ship waves, and time-harmonic ship waves are considered subsequently.

### Special cases of rigid and soft free-surface planes

The free-surface contribution (17) is null if the boundary condition at  $\Sigma^F$  is  $w = 0$  and the function  $\mathcal{H}$  in (12a) is chosen as  $\mathcal{H} = 0$ , so that the Green function satisfies the condition  $G_\zeta = 0$  at  $\zeta = 0$ . The free-surface contribution (17) is also null if the boundary condition at  $\Sigma^F$  is  $\phi = 0$ , which implies  $u = 0 = v$ , and the function  $H$  in (12a) is taken as  $H = 0$ , which yields  $G = 0$  at  $\zeta = 0$ . Thus, the potential  $\phi$  and the velocity  $\vec{u}$  are defined by (14) as

$$\phi = \begin{pmatrix} \phi^R + \phi_*^R \\ \phi^R - \phi_*^R \end{pmatrix} \quad \text{and} \quad \vec{u} = \begin{pmatrix} \vec{u}^R + \vec{u}_*^R \\ \vec{u}^R - \vec{u}_*^R \end{pmatrix} \quad \text{if} \quad \begin{pmatrix} w = 0 \\ \phi = 0 \end{pmatrix} \quad \text{at} \quad z = 0 \quad (18)$$

Here,  $\phi^R$ ,  $\vec{u}^R$  and  $\phi_*^R$ ,  $\vec{u}_*^R$  are given by (15) where the boundary surface  $\Sigma$  is taken as the hull+wake surface  $\Sigma^{HW}$ .

### Wave diffraction-radiation without forward speed

The free-surface contribution (17) is now considered for the free-surface boundary condition

$$w = f^2 \phi + p \quad \text{at} \quad z = 0 \quad (19)$$

associated with diffraction-radiation of time-harmonic water waves without forward speed. Here,  $f = \omega \sqrt{L/g}$  is the nondimensional wave frequency, and  $p$  stands for a nonhomogeneous forcing

term. E.g.,  $p$  may account for a pressure distribution at the free surface. The corresponding Green function  $G = G^S + G^F$  satisfies the free-surface boundary condition

$$G_\zeta = f^2 G \quad \text{at} \quad \zeta = 0 \quad (20a)$$

The decompositions (12a) then yield  $\mathcal{H}_\zeta = f^2 H$  at  $\zeta = 0$ . This relation also holds for  $\zeta < 0$  since  $\mathcal{H}$  and  $H$  are functions of  $z + \zeta$ . We then have

$$f^2 H = \mathcal{H}_\zeta = \mathcal{H}_z \quad (20b)$$

It follows that the Green function also satisfies the condition

$$G_z = f^2 G \quad \text{at} \quad z = 0 \quad (20c)$$

By using (19), (20b), and the Laplace equation  $\nabla^2 \mathcal{H} = 0$  in (17) we can obtain

$$4\pi \begin{pmatrix} \phi \\ u \\ v \\ w \end{pmatrix} = \int_{\Sigma^F} d\mathcal{A} \begin{pmatrix} -p H \\ (\phi \mathcal{H}_z)_x + p H_x \\ (\phi \mathcal{H}_z)_y + p H_y \\ (\phi \mathcal{H}_x)_x + (\phi \mathcal{H}_y)_y - p H_z \end{pmatrix}$$

The flow representation (14) then yields

$$\begin{pmatrix} \phi = \phi^R + \phi_*^R + \phi^{\mathcal{H}} - \phi^p \\ \vec{u} = \vec{u}^R + \vec{u}_*^R + \vec{u}^{\mathcal{H}} + \vec{u}_f^{\mathcal{H}} - \vec{u}^p \end{pmatrix} \quad (21a)$$

The components  $\phi^R$ ,  $\vec{u}^R$  and  $\phi_*^R$ ,  $\vec{u}_*^R$  in (21a) are given by (15) where the boundary surface  $\Sigma$  is restricted to the hull+wake surface  $\Sigma^{HW}$ . The component  $\phi^{\mathcal{H}}$ ,  $\vec{u}^{\mathcal{H}}$  is given by (16) with  $\Sigma = \Sigma^{HW}$  and  $\mathcal{F} = \mathcal{H}$ . Stokes' theorem shows that the component  $\vec{u}_f^{\mathcal{H}}$ , which accounts for the free-surface contribution (17) if  $p = 0$ , is given by

$$4\pi \begin{pmatrix} u_f^{\mathcal{H}} \\ v_f^{\mathcal{H}} \\ w_f^{\mathcal{H}} \end{pmatrix} = \int_{\Gamma^{HW}} d\mathcal{L} \phi \begin{pmatrix} t^y \mathcal{H}_z \\ -t^x \mathcal{H}_z \\ t^y \mathcal{H}_x - t^x \mathcal{H}_y \end{pmatrix} \quad (21b)$$

Finally, the component  $\phi^p$ ,  $\vec{u}^p$  associated with the nonhomogeneous term  $p$  in the free-surface condition (19) is given by

$$4\pi \begin{pmatrix} \phi^p \\ u^p \\ v^p \\ w^p \end{pmatrix} = \int_{\Sigma^F} d\mathcal{A} p \begin{pmatrix} H \\ -H_x \\ -H_y \\ H_z \end{pmatrix} = \int_{\Sigma^F} d\mathcal{A} p \begin{pmatrix} H \\ H_\xi \\ H_\eta \\ H_\zeta \end{pmatrix} \quad (21c)$$

(8) and (12) yield

$$\begin{Bmatrix} \phi^p \\ \vec{u}^p \end{Bmatrix} = \int_{\Sigma^F} d\mathcal{A} \, p \begin{Bmatrix} G \\ \nabla_\xi G \end{Bmatrix}$$

The representation (21a) consists of two components, which define the potential  $\phi$  and the velocity  $\vec{u}$ . The potential component of (21a) is a classical representation that defines  $\phi$  in the flow domain in terms of the values of  $\vec{u} \cdot \vec{n}$  and  $\phi$  at the boundary surface  $\Sigma^{HW}$ . This representation involves both the Green function and its first derivatives. The velocity component of (21a) is a new boundary-integral representation that defines  $\vec{u}$  in the flow domain in terms of  $\vec{u}$  at the boundary surface  $\Sigma^{HW}$  and  $\phi \equiv (w-p)/f^2$  at the boundary curve  $\Gamma^{HW}$ . The representation of  $\vec{u}$  only involves  $\nabla G$ , i.e. it does not involve either the Green function  $G$  or higher-than-first derivatives of  $G$ . The velocity component of (21a) involves the line integral (21b) around the boundary curve  $\Gamma^{HW}$ ; the potential component of (21a) does not involve such a line integral.

### Steady ship waves

The free-surface contribution (17) is now considered for the Kelvin free-surface condition

$$w + F^2 u_x = p \quad \text{at} \quad z = 0 \quad (22)$$

associated with steady ship waves. Here,  $F = \mathcal{U}/\sqrt{gL}$  is the Froude number, and  $p$  stands for a nonhomogeneous forcing term as in (19). The Kelvin source potential  $G = G^S + G^F$  satisfies the free-surface boundary condition

$$G_\zeta + F^2 G_{\xi\xi} = 0 \quad \text{at} \quad \zeta = 0 \quad (23a)$$

The decompositions (12a) then yield  $\mathcal{H}_\zeta + F^2 H_{\xi\xi} = 0$  at  $\zeta = 0$ . This relation also holds for  $\zeta < 0$  because  $H$  and  $\mathcal{H}$  are functions of  $z + \zeta$ . We then have

$$\mathcal{H}_\zeta = -F^2 H_{\xi\xi} \quad (23b)$$

It follows that the functions  $H$  and  $\mathcal{H}$  and the Green function  $G$  also satisfy the relations

$$\mathcal{H}_z = -F^2 H_{xx} \quad \mathcal{H} = -F^2 H_{xx}^z \quad (23c)$$

$$G_z + F^2 G_{xx} = 0 \quad \text{at} \quad z = 0 \quad (23d)$$

The function  $H^z$  in (23c) is defined as

$$H^z(x - \xi, y - \eta, z + \zeta) = \int_{-\infty}^z dt \, H(x - \xi, y - \eta, t + \zeta) \quad (24)$$

By using the free-surface condition (22), the relations (23c), the irrotationality condition  $v_x = u_y$  and the Laplace equation  $\nabla^2 H_x^z = 0$  in (17) we can obtain

$$4\pi \begin{Bmatrix} \phi \\ u \\ v \\ w \end{Bmatrix} = -F^2 \int_{\Sigma^F} d\mathcal{A} \vec{V}^F - 4\pi \begin{Bmatrix} \phi^p \\ u^p \\ v^p \\ w^p \end{Bmatrix}$$

where  $\phi^p$  and  $\vec{u}^p$  are given by (21c) and  $\vec{V}_F$  is the vector defined as

$$\vec{V}^F = \begin{Bmatrix} (\phi H_x - u H)_x \\ (u H_x)_x \\ (v H_x)_x - (u H_x)_y + (u H_y)_x \\ (v H_{xy}^z)_x - (u H_{xy}^z)_y - (u H_z)_x \end{Bmatrix} \quad (25)$$

The flow representation (14) then yields

$$\begin{Bmatrix} \phi = \phi^R - \phi_*^R + \phi^H + \phi_F^H - \phi^p \\ \vec{u} = \vec{u}^R - \vec{u}_*^R + \vec{u}^H + \vec{u}_F^H - \vec{u}^p \end{Bmatrix} \quad (26a)$$

The components  $\phi^R$ ,  $\vec{u}^R$  and  $\phi_*^R$ ,  $\vec{u}_*^R$  in (26a) are given by (15) where the boundary surface  $\Sigma$  is restricted to the hull+wake surface  $\Sigma^{HW}$ . The component  $\phi^H$ ,  $\vec{u}^H$  is given by (16) with  $\Sigma = \Sigma^{HW}$  and  $\mathcal{F} = H$ . Stokes' theorem shows that the component  $\phi_F^H$ ,  $\vec{u}_F^H$  associated with the free-surface contribution (17) is given by

$$4\pi \begin{Bmatrix} -\phi_F^H \\ u_F^H \\ v_F^H \\ -w_F^H \end{Bmatrix} = -F^2 \int_{\Gamma^{HW}} d\mathcal{L} \begin{Bmatrix} u t^y H - \phi t^y H_x \\ u t^y H_x \\ u t^y H_y + \vec{u} \cdot \vec{t} H_x \\ u t^y H_z - \vec{u} \cdot \vec{t} H_{xy}^z \end{Bmatrix}$$

The identity  $u = (\vec{u} \cdot \vec{t}) t^x - (\vec{u} \cdot \vec{\nu}) t^y$ , where  $\vec{t} = (t^x, t^y)$  and  $\vec{\nu} = (-t^y, t^x)$  are unit vectors tangent and normal to the boundary curve  $\Gamma^{HW}$ , yields the alternative representation

$$4\pi \begin{Bmatrix} -\phi_F^H \\ u_F^H \\ v_F^H \\ -w_F^H \end{Bmatrix} = F^2 \int_{\Gamma^{HW}} d\mathcal{L} \begin{Bmatrix} (\vec{u} \cdot \vec{\nu}) (t^y)^2 H - (\vec{u} \cdot \vec{t}) t^x t^y H + \phi t^y H_x \\ (\vec{u} \cdot \vec{\nu}) (t^y)^2 H_x - (\vec{u} \cdot \vec{t}) t^x t^y H_x \\ (\vec{u} \cdot \vec{\nu}) (t^y)^2 H_y - (\vec{u} \cdot \vec{t}) (t^x t^y H_y + H_x) \\ (\vec{u} \cdot \vec{\nu}) (t^y)^2 H_z - (\vec{u} \cdot \vec{t}) (t^x t^y H_z - H_{xy}^z) \end{Bmatrix} \quad (26b)$$

Both the potential and velocity components of (26a) involve a line integral around  $\Gamma^{HW}$ . The potential component is a classical representation that defines  $\phi$  in the flow domain in terms of the

values of  $\vec{u} \cdot \vec{n}$  and  $\phi$  at  $\Sigma^{HW}$  and the values of  $\vec{u}$  and  $\phi$  at  $\Gamma^{HW}$ . This representation involves  $G$  and  $\nabla G$ . The velocity component of (26a) is a new representation that defines  $\vec{u}$  in the flow domain in terms of the boundary values of the normal velocity  $\vec{u} \cdot \vec{n}$ ,  $\vec{u} \cdot \vec{\nu}$  and the tangential velocity  $\vec{u} \times \vec{n}$ ,  $\vec{u} \cdot \vec{t}$ . Thus, the velocity representation does not involve  $\phi$ . This representation involves  $\nabla G$  and the function  $H_{xy}^z$  defined by (24), which is comparable to a first derivative ( $H_{xy}^z$  and  $\nabla_x H$  have similar behaviors in the near field and the far field).

### Time-harmonic ship waves

The free-surface contribution (17) is now considered for the free-surface condition

$$w - f^2 \phi + i 2\tau u + F^2 u_x = p \quad \text{at} \quad z = 0 \quad (27)$$

corresponding to wave diffraction-radiation with forward speed. Here,  $f$  and  $F$  are the nondimensional wave frequency and the Froude number, and  $\tau = fF$ . The ship-motion Green function  $G = G^S + G^F$  satisfies the free-surface boundary condition

$$G_\zeta - f^2 G + i 2\tau G_\xi + F^2 G_{\xi\xi} = 0 \quad \text{at} \quad \zeta = 0 \quad (28a)$$

The decompositions (12a) yield  $\mathcal{H}_\zeta - f^2 H + i 2\tau H_\xi + F^2 H_{\xi\xi} = 0$  at  $\zeta = 0$ . This relation also holds for  $\zeta < 0$  since  $H$  and  $\mathcal{H}$  are functions of  $z + \zeta$ . We then have

$$\mathcal{H}_\zeta = f^2 H - i 2\tau H_\xi - F^2 H_{\xi\xi} \quad (28b)$$

It follows that the functions  $H$  and  $\mathcal{H}$  and the Green function  $G$  also satisfy the relations

$$\mathcal{H}_z = f^2 H + i 2\tau H_x - F^2 H_{xx} \quad \mathcal{H} = f^2 H^z + i 2\tau H_x^z - F^2 H_{xx}^z \quad (28c)$$

$$G_z - f^2 G - i 2\tau G_x + F^2 G_{xx} = 0 \quad \text{at} \quad z = 0 \quad (28d)$$

$H^z$  in (28c) is defined by (24). By using the free-surface condition (27), the relations (28c), the irrotationality condition  $v_x = u_y$  and the Laplace equation  $\nabla^2 H^z = 0$  in (17) we can obtain

$$4\pi \begin{pmatrix} \phi \\ u \\ v \\ w \end{pmatrix} = \int_{\Sigma^F} dA (f^2 \vec{V}_f + i 2\tau \vec{V}_\tau - F^2 \vec{V}_F) - 4\pi \begin{pmatrix} \phi^p \\ u^p \\ v^p \\ w^p \end{pmatrix}$$

where the component  $\phi^p$ ,  $\vec{u}^p$  is defined by (21c) and the vectors  $\vec{V}_F$ ,  $\vec{V}_f$ ,  $\vec{V}_\tau$  are given by (25) and

$$\vec{V}_f = \begin{Bmatrix} 0 \\ (\phi H)_x \\ (\phi H)_y \\ (\phi H_x^z)_x + (\phi H_y^z)_y \end{Bmatrix} \quad \vec{V}_\tau = \begin{Bmatrix} (\phi H)_x \\ 0 \\ (v H)_x - (u H)_y \\ (v H_y^z)_x - (u H_y^z)_y \end{Bmatrix}$$

The flow representation (14) yields

$$\begin{Bmatrix} \phi = \phi^R - \phi_*^R + \phi^H + \phi_F^H + \phi_\tau^H - \phi^p \\ \vec{u} = \vec{u}^R - \vec{u}_*^R + \vec{u}^H + \vec{u}_F^H + \vec{u}_\tau^H + \vec{u}_f^H - \vec{u}^p \end{Bmatrix} \quad (29a)$$

The components  $\phi^R, \vec{u}^R$  and  $\phi_*^R, \vec{u}_*^R$  in (29a) are given by (15) with  $\Sigma = \Sigma^{HW}$ . The component  $\phi^H, \vec{u}^H$  is given by (16) with  $\Sigma = \Sigma^{HW}$  and  $\mathcal{F} = H$ . The component  $\phi_F^H, \vec{u}_F^H$  is given by (26b). Stokes' theorem shows that the components  $\phi_\tau^H, \vec{u}_\tau^H$  and  $\vec{u}_f^H$  are given by

$$4\pi \begin{Bmatrix} \phi_\tau^H \\ u_\tau^H \\ v_\tau^H \\ w_\tau^H \end{Bmatrix} = i 2\tau \int_{\Gamma^{HW}} d\mathcal{L} \begin{Bmatrix} \phi t^y H \\ 0 \\ \vec{u} \cdot \vec{t} H \\ \vec{u} \cdot \vec{t} H_y^z \end{Bmatrix} \quad (29b)$$

$$4\pi \begin{Bmatrix} u_f^H \\ v_f^H \\ w_f^H \end{Bmatrix} = f^2 \int_{\Gamma^{HW}} d\mathcal{L} \phi \begin{Bmatrix} t^y H \\ -t^x H \\ t^y H_x^z - t^x H_y^z \end{Bmatrix} \quad (29c)$$

In the special cases  $f=0$  and  $F=0$ , (29) is identical to (26) and (21) as one expects.

### Fourier component in free-surface Green functions

Free-surface effects in the potential and velocity representations (21a), (26a), (29a) are defined in terms of the components  $\mathcal{H}$  or  $H$  in the alternative decompositions (12a) of the free-surface component  $G^F$  in the representation (8a) of the Green functions associated with the free-surface conditions

$$\begin{Bmatrix} w - f^2 \phi = p \\ w + F^2 u_x = p \\ w - f^2 \phi + i 2\tau u + F^2 u_x = p \end{Bmatrix} \quad \text{at } z = 0 \quad (30)$$

for diffraction-radiation of time-harmonic waves without forward speed, steady ship waves, and diffraction-radiation of time-harmonic waves with forward speed. The free-surface components  $\mathcal{H}$



and  $H$  are given by the Fourier superposition of elementary waves

$$\mathcal{F} = \lim_{\varepsilon \rightarrow +0} \frac{1}{\pi} \int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} d\alpha \frac{A}{D_\varepsilon} \mathcal{E} \mathcal{E}^* \quad (31a)$$

with

$$\mathcal{F} = \begin{Bmatrix} \mathcal{H} \\ H \end{Bmatrix} \quad A = \begin{Bmatrix} f^2/k \\ 1 \end{Bmatrix} \quad \text{if } \begin{Bmatrix} F = 0 \\ F > 0 \end{Bmatrix} \quad (31b)$$

$$\begin{Bmatrix} D_\varepsilon = D + i\varepsilon D' \\ D = (f - F\alpha)^2 - k \\ D' = f - F\alpha \end{Bmatrix} \quad (31c)$$

$$\begin{Bmatrix} \mathcal{E} = \exp[kz + i(\alpha x + \beta y)] \\ \mathcal{E}^* = \exp[k\zeta - i(\alpha \xi + \beta \eta)] \end{Bmatrix} \quad (31d)$$

In the special case  $F = 0$ , i.e. for wave diffraction-radiation without forward speed, (31b) yields  $A = 1$  if  $k = f^2$ , i.e. at the dispersion curve defined by the dispersion relation  $D = 0$ . The flow related to the Fourier component  $\mathcal{F}$  defined by (31) can be analyzed simply and effectively using the Fourier-Kochin approach expounded in Noblesse and Yang [1]. This approach is used below.

### Fourier-Kochin representation of free-surface effects

By substituting (31a) into (16), (21b), (26b), (29b), (29c) and exchanging the order in which the integrations with respect to the Fourier variables  $(\alpha, \beta)$  and the space variables  $(x, y, z)$  are performed, we can express the free-surface components in (21a) and (29a) as

$$4\pi \begin{Bmatrix} \phi^{\mathcal{H}} \\ u^{\mathcal{H}} + u_f^{\mathcal{H}} \\ v^{\mathcal{H}} + v_f^{\mathcal{H}} \\ w^{\mathcal{H}} + w_f^{\mathcal{H}} \end{Bmatrix} = \lim_{\varepsilon \rightarrow +0} \frac{1}{\pi} \int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} d\alpha \frac{f^2 \mathcal{E}^*}{k D_\varepsilon} \begin{Bmatrix} S^\phi \\ S^u + k S_f^u \\ S^v + k S_f^v \\ S^w + k S_f^w \end{Bmatrix} \quad (32a)$$

$$4\pi \begin{Bmatrix} \phi^H + \phi_F^H + \phi_\tau^H \\ u^H + u_F^H + u_f^H \\ v^H + v_F^H + v_f^H + v_\tau^H \\ w^H + w_F^H + w_f^H + w_\tau^H \end{Bmatrix} = \lim_{\varepsilon \rightarrow +0} \frac{1}{\pi} \int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} d\alpha \frac{\mathcal{E}^*}{D_\varepsilon} \begin{Bmatrix} S^\phi + F^2 S_F^\phi + 2\tau S_\tau^\phi \\ S^u + F^2 S_F^u + f^2 S_f^u \\ S^v + F^2 S_F^v + f^2 S_f^v + 2\tau S_\tau^v \\ S^w + F^2 S_F^w + f^2 S_f^w + 2\tau S_\tau^w \end{Bmatrix} \quad (32b)$$

if  $F = 0$  or  $F > 0$ , respectively. The Fourier components  $\phi^H$ ,  $\vec{u}^H$  and  $\phi_F^H$ ,  $\vec{u}_F^H$  in the representations (26a) and (29a) are identical. Indeed, the Fourier-Kochin representation of steady ship

waves is a particular case of the Fourier-Kochin representation of time-harmonic ship waves, and is obtained by setting  $f = 0$  and  $\tau = 0$  in (32b).

The spectrum functions  $S^\phi, S^u, S^v, S^w$  corresponding to the components  $\phi^{\mathcal{H}}, \vec{u}^{\mathcal{H}}$  and  $\phi^H, \vec{u}^H$  defined by (16) are given by

$$\begin{pmatrix} S^\phi \\ S^u \\ S^v \\ S^w \end{pmatrix} = \int_{\Sigma^{HW}} d\mathcal{A} \begin{pmatrix} \vec{u} \cdot \vec{n} - (i\alpha n^x + i\beta n^y + k n^z) \phi \\ -i\alpha \vec{u} \cdot \vec{n} - i\beta (\vec{u} \times \vec{n})^z + k (\vec{u} \times \vec{n})^y \\ -i\beta \vec{u} \cdot \vec{n} - k (\vec{u} \times \vec{n})^x + i\alpha (\vec{u} \times \vec{n})^z \\ k \vec{u} \cdot \vec{n} + i\alpha (\vec{u} \times \vec{n})^y - i\beta (\vec{u} \times \vec{n})^x \end{pmatrix} \mathcal{E} \quad (33a)$$

The spectrum functions  $S_f^u, S_f^v, S_f^w$  corresponding to the components  $\vec{u}_f^{\mathcal{H}}$  and  $\vec{u}_f^H$  defined by (21b) and (29c) are given by

$$\begin{pmatrix} S_f^u \\ S_f^v \\ S_f^w \end{pmatrix} = \int_{\Gamma^{HW}} d\mathcal{L} \begin{pmatrix} t^y \\ -t^x \\ i(\alpha t^y - \beta t^x)/k \end{pmatrix} \phi \mathcal{E}_0 \quad (33b)$$

with  $\mathcal{E}_0 = \exp[i(\alpha x + \beta y)]$ . The spectrum functions  $S_F^\phi, S_F^u, S_F^v, S_F^w$  corresponding to the component  $\phi_F^H, \vec{u}_F^H$  defined by (26b) are given by

$$\begin{pmatrix} S_F^\phi \\ S_F^u \\ S_F^v \\ S_F^w \end{pmatrix} = \int_{\Gamma^{HW}} d\mathcal{L} \begin{pmatrix} -(t^y)^2 \vec{u} \cdot \vec{v} + t^x t^y \vec{u} \cdot \vec{t} - i\alpha t^y \phi \\ i\alpha (t^y)^2 \vec{u} \cdot \vec{v} - i\alpha t^x t^y \vec{u} \cdot \vec{t} \\ i\beta (t^y)^2 \vec{u} \cdot \vec{v} - i(\beta t^x t^y + \alpha) \vec{u} \cdot \vec{t} \\ -k (t^y)^2 \vec{u} \cdot \vec{v} + (k t^x t^y + \alpha \beta/k) \vec{u} \cdot \vec{t} \end{pmatrix} \mathcal{E}_0 \quad (33c)$$

The spectrum functions  $S_\tau^\phi, S_\tau^v, S_\tau^w$  corresponding to the component  $\phi_\tau^H, \vec{u}_\tau^H$  defined by (29b) are given by

$$\begin{pmatrix} S_\tau^\phi \\ S_\tau^v \\ S_\tau^w \end{pmatrix} = \int_{\Gamma^{HW}} d\mathcal{L} \begin{pmatrix} i t^y \phi \\ i \vec{u} \cdot \vec{t} \\ -(\beta/k) \vec{u} \cdot \vec{t} \end{pmatrix} \mathcal{E}_0 \quad (33d)$$

Similarly, (31a) and (21c) show that the component  $\phi^p, \vec{u}^p$  in (21a), (26a), (29a) is given by the Fourier-Kochin representation

$$4\pi \begin{pmatrix} \phi^p \\ u^p \\ v^p \\ w^p \end{pmatrix} = \lim_{\epsilon \rightarrow +0} \frac{1}{\pi} \int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} d\alpha \frac{\mathcal{E}^*}{D_\epsilon} \begin{pmatrix} 1 \\ -i\alpha \\ -i\beta \\ k \end{pmatrix} S^p \quad (34)$$

with  $S^p$  given by (35c).

The next section shows that the 15 spectrum functions in (32) can be expressed in terms of 4 basic spectrum functions  $S$ ,  $S_*^\phi$ ,  $S_*^t$ ,  $S^F$ . These 4 basic spectrum functions and the spectrum function  $S^p$  in (34) are given by

$$S = \int_{\Sigma^{HW}} dA [\vec{u} \cdot \vec{n} + i \frac{\alpha}{k} (\vec{u} \times \vec{n})^y - i \frac{\beta}{k} (\vec{u} \times \vec{n})^x] \mathcal{E} \quad (35a)$$

$$\begin{pmatrix} S_*^\phi \\ S_*^t \\ S^F \end{pmatrix} = \int_{\Gamma^{HW}} d\mathcal{L} \begin{pmatrix} i \phi (\alpha t^y - \beta t^x) / k \\ \vec{u} \cdot \vec{t} \\ (t^x t^y + \alpha \beta / k^2) \vec{u} \cdot \vec{t} - (t^y)^2 \vec{u} \cdot \vec{v} \end{pmatrix} \mathcal{E}_0 \quad (35b)$$

$$S^p = \int_{\Sigma^F} dA p \mathcal{E}_0 \quad (35c)$$

$$\text{with} \quad \mathcal{E} = e^{kz + i(\alpha x + \beta y)} \quad \mathcal{E}_0 = e^{i(\alpha x + \beta y)} \quad (35d)$$

The spectrum function  $S$ , the spectrum functions  $S_*^\phi$ ,  $S_*^t$ ,  $S^F$ , and the spectrum function  $S^p$  are respectively given by distributions of elementary waves over the boundary surface  $\Sigma^{HW}$ , the boundary curve  $\Gamma^{HW}$ , and the mean free surface  $\Sigma^F$ . The spectrum functions  $S$ ,  $S_*^t$ ,  $S^F$  are defined in terms of the normal components  $\vec{u} \cdot \vec{n}$ ,  $\vec{u} \cdot \vec{v}$  and the tangential components  $\vec{u} \times \vec{n}$ ,  $\vec{u} \cdot \vec{t}$  of the velocity  $\vec{u}$  at the boundary surface  $\Sigma^{HW}$  and the boundary curve  $\Gamma^{HW}$ . The spectrum function  $S_*^\phi$  is defined in terms of the potential  $\phi$ .

### Transformations of spectrum functions in Fourier-Kochin representation

Useful transformations of the spectrum functions in (32) are given below. These transformations are performed in two steps. The first step is the transformation (37), which expresses the spectrum functions  $S^\phi$ ,  $S^u$ ,  $S^v$ ,  $S^w$  given by (33a) in terms of the three spectrum functions  $S$ ,  $S_*^\phi$ ,  $S_*^t$ . The spectrum function  $S^\phi$  associated with the potentials  $\phi^{\mathcal{H}}$  and  $\phi^H$  in (32) has already been considered in Noblesse and Yang [1]. Expressions (33), (29), (36), (34) in this previous study, where the vector  $\vec{n}$  normal to the boundary surface  $\Sigma^{HW}$  points outside the flow domain (instead of inside in the present study), yield the important basic transformation

$$\begin{aligned} & - \int_{\Sigma^{HW}} dA (i \alpha n^x + i \beta n^y + k n^z) \phi \mathcal{E} = \\ & i \int_{\Sigma^{HW}} dA \left[ \frac{\alpha}{k} (\vec{u} \times \vec{n})^y - \frac{\beta}{k} (\vec{u} \times \vec{n})^x \right] \mathcal{E} + S_*^\phi \end{aligned}$$

with  $S_*^\phi$  given by (35b). The spectrum function  $S^\phi$  in (33a) can then be expressed as

$$S^\phi = S + S_*^\phi \quad (36a)$$

with  $S$  given by (35a). Expressions (33a) and (35a) immediately yield

$$S^w = k S \quad (36b)$$

Section A of Appendix 2 shows that (33a) and (35a) also yield

$$\begin{cases} S^u + i \alpha S = -i \beta S_*^t / k \\ S^v + i \beta S = i \alpha S_*^t / k \end{cases} \quad (36c)$$

Expressions (36) yield

$$\begin{pmatrix} S^\phi \\ S^u \\ S^v \\ S^w \end{pmatrix} = \begin{pmatrix} 1 \\ -i \alpha \\ -i \beta \\ k \end{pmatrix} S + \vec{S}_* \quad (37a)$$

with

$$\vec{S}_* = \begin{pmatrix} S_*^\phi \\ -i \beta S_*^t / k \\ i \alpha S_*^t / k \\ 0 \end{pmatrix} \quad (37b)$$

These expressions for the spectrum functions  $S^\phi, S^u, S^v, S^w$  — with  $S, S_*^\phi, S_*^t$  given by (35) — are now used to modify the spectrum functions in the Fourier-Kochin representations (32).

The spectrum functions in (32a), corresponding to diffraction-radiation of time-harmonic waves without forward speed, can be expressed as

$$\begin{pmatrix} S^\phi \\ S^u + k S_f^u \\ S^v + k S_f^v \\ S^w + k S_f^w \end{pmatrix} = \begin{pmatrix} 1 \\ -i \alpha \\ -i \beta \\ k \end{pmatrix} (S + S_*^\phi) \quad (38a)$$

with  $S$  and  $S_*^\phi$  given by (35). The expression for the spectrum function  $S^\phi$  follows immediately from (37). The expression for the spectrum function related to  $w$  also follows immediately from (37), (33b) and (35b). The expressions for the spectrum functions related to  $u$  and  $v$  are verified in Section B of Appendix 2. In the special case  $f = 0$ , corresponding to steady ship waves, the

spectrum functions in (32b) can be expressed as

$$\begin{pmatrix} S^\phi + F^2 S_F^\phi \\ S^u + F^2 S_F^u \\ S^v + F^2 S_F^v \\ S^w + F^2 S_F^w \end{pmatrix} = \begin{pmatrix} 1 \\ -i\alpha \\ -i\beta \\ k \end{pmatrix} (S + F^2 S^F) - \frac{D}{k} \vec{S}_* \quad (38b)$$

Here,  $S$  and  $S^F$  are given by (35),  $D = F^2 \alpha^2 - k$ , and  $\vec{S}_*$  is defined by (37b) and (35b). The expression for the spectrum function related to  $w$  follows immediately from (37) and (33c). The expressions for the spectrum functions related to  $\phi$ ,  $u$  and  $v$  are verified in Section C of Appendix 2. In the general case  $fF > 0$ , corresponding to time-harmonic ship waves, the spectrum functions in (32b) can be expressed as

$$\begin{pmatrix} S^\phi + F^2 S_F^\phi + 2\tau S_\tau^\phi \\ S^u + F^2 S_F^u + f^2 S_f^u \\ S^v + F^2 S_F^v + f^2 S_f^v + 2\tau S_\tau^v \\ S^w + F^2 S_F^w + f^2 S_f^w + 2\tau S_\tau^w \end{pmatrix} = \begin{pmatrix} 1 \\ -i\alpha \\ -i\beta \\ k \end{pmatrix} \hat{S} - \frac{D}{k} \vec{S}_* \quad (38c)$$

where  $D = (F\alpha - f)^2 - k$ ,  $\vec{S}_*$  is given by (37b) and (35b), and  $\hat{S}$  is defined as

$$\hat{S} = S + \frac{f^2}{k} S_*^\phi + F^2 S^F - \frac{2\tau}{k} \frac{\beta}{k} S_*^t$$

Expressions (38c) are verified in Section D of Appendix 2. These expressions are identical to (38b) in the special case  $f = 0$ .

### Fourier-Kochin representation of waves

Noblesse and Chen [5,6] and Noblesse et al. [7] show that double Fourier integral representations of free-surface effects like (32) and (34) can be expressed as sums of wave and local components:

$$\begin{pmatrix} \phi^{\mathcal{H}} - \phi^p \\ \vec{u}^{\mathcal{H}} + \vec{u}_f^{\mathcal{H}} - \vec{u}^p \end{pmatrix} = \begin{pmatrix} \phi^W + \phi^L \\ \vec{u}^W + \vec{u}^L \end{pmatrix} \quad (39a)$$

$$\begin{pmatrix} \phi^H + \phi_F^H + \phi_\tau^H - \phi^p \\ \vec{u}^H + \vec{u}_F^H + \vec{u}_\tau^H + \vec{u}_f^H - \vec{u}^p \end{pmatrix} = \begin{pmatrix} \phi^W + \phi^L \\ \vec{u}^W + \vec{u}^L \end{pmatrix} \quad (39b)$$

The most important result given in [5-7] is a remarkably simple expression for the wave component  $\phi^W$ ,  $\vec{u}^W$ . This expression defines the wave components included in the double Fourier integrals (32)

and (34) in terms of single Fourier integrals along the dispersion curves defined by the dispersion relation  $D = 0$ . It follows that the second components on the right of (38b) and (38c), which involve the dispersion function  $D$  as a factor, do not contribute to the waves contained in the Fourier representations (32). Specifically, the wave component  $\phi^W, \vec{u}^W$  is given by

$$4\pi \begin{pmatrix} \phi^W \\ u^W \\ v^W \\ w^W \end{pmatrix} = -i \sum_{D=0} \int_{D=0} ds \frac{\text{sign}(D') + \Theta}{\sqrt{D_\alpha^2 + D_\beta^2}} \begin{pmatrix} 1 \\ -i\alpha \\ -i\beta \\ k \end{pmatrix} S^W \mathcal{E}^* \quad (40a)$$

with  $\mathcal{E}^* = e^{\zeta k - i(\xi\alpha + \eta\beta)}$  as given by (31d). The identity  $f^2 = k$  at a dispersion curve  $D = 0$ , which follows from (31c), was used in (40a).  $(D_\alpha, D_\beta)$  are the derivatives of the dispersion function  $D$  with respect to the Fourier variables  $(\alpha, \beta)$ .  $D$  and  $D'$  are defined by (31c) as  $D = (f - F\alpha)^2 - k$  and  $D' = f - F\alpha$ . The function  $\Theta$  is the error function

$$\Theta = \text{erf}\left(\frac{k(\xi D_\alpha + \eta D_\beta)}{\sigma \sqrt{D_\alpha^2 + D_\beta^2}}\right)$$

where  $\sigma \approx 1$  is a positive real number. The wave-spectrum function  $S^W$  in (40a) is defined by (32), (34), (38) as

$$S^W = \begin{cases} S + S_*^\phi \\ S + F^2 S^F \\ S + \frac{f^2}{k} S_*^\phi + F^2 S^F - \frac{2\tau}{k} \frac{\beta}{k} S_*^t \end{cases} - S^p \quad \text{if} \quad \begin{cases} F = 0 \\ f = 0 \\ fF > 0 \end{cases} \quad (40b)$$

Thus, waves are defined in terms of the spectrum functions  $S^W$  and  $S^p$  given by (40b) and (35). In the special case  $f = 0$ , i.e. for steady flow, the spectrum function  $S^W$  is given in terms of the spectrum functions  $S$  and  $S^F$ , which do not involve the potential  $\phi$  and are defined directly in terms of the velocity  $\vec{u}$ . If  $f > 0$ ,  $S^W$  involves the spectrum function  $S_*^\phi$  which is defined in terms of  $\phi$ , although the free-surface boundary condition (30) can be used to express  $\phi$  in terms of  $\vec{u}$ .

The representation (40a) of the wave component  $\phi^W, \vec{u}^W$  agrees with the relation

$$\vec{u}^W(\vec{\xi}) = \nabla_\xi \phi^W(\vec{\xi})$$

used in Noblesse et al. [8] to determine the velocity  $\vec{u}^W$  from the representation of the potential  $\phi^W$  given in Noblesse and Yang [1]. Thus, the velocity representation obtained in the present study and the potential representation given in Noblesse and Yang [1] yield identical waves. This agreement needed to be verified since the relations (38) are not a-priori obvious.

## Conclusion

A new fundamental mathematical representation of free-surface flows has been given. This flow representation, called velocity representation, is given by (14)-(16) for generic free-surface flows associated with a Green function  $G$  of the type defined by (8) and (12). The velocity representation, and the classical potential representation, are given by (18), (21), (26), (29) with (15) and (16) for the four classes of flows corresponding to the particular cases of (i) an infinitely rigid or soft free-surface plane, (ii) diffraction-radiation of time-harmonic water waves without forward speed, (iii) steady ship waves, and (iv) time-harmonic ship waves (diffraction-radiation with forward speed). The potential and velocity representations, given here for deep water, can be extended to uniform finite water depth, and indeed can be extended to a broader class of problems involving dispersive waves that propagate in the presence of a planar boundary.

The velocity representation defines the velocity  $\vec{u}$  inside a flow domain in terms of source and vortex distributions with strength equal to the normal and tangential components of the velocity  $\vec{u}$  at the boundary surface. Thus, the velocity representation does not involve the velocity potential. This property is an important difference between the velocity representation and the classical potential representation, which defines the velocity potential  $\phi$  within a potential-flow region in terms of the potential  $\phi$  and its normal derivative  $\partial\phi/\partial n$  at the boundary surface. The velocity representation can then be used to couple a nearfield flow calculation method based on the Euler or RANS equations, for which a velocity potential cannot be defined, and a farfield potential flow representation. Another notable difference between the potential and velocity representations is that the velocity representation defines  $\vec{u}$  in terms of first derivatives of the Green function  $G$ , whereas  $\vec{u}$  can only be obtained from the potential representation via either numerical or analytical differentiation of  $\phi$ . Analytical differentiation of the potential representation involves second-order derivatives of  $G$ , as shown in (3c).

The velocity representation provides a new mathematical basis for computing free-surface flows about nonlifting and lifting bodies using free-surface Green functions. In particular, this boundary-integral representation has been used here together with the Fourier-Kochin approach expounded in Noblesse and Yang [1] to obtain analytical representations, given by (40) and (35), of the waves generated by an arbitrary boundary velocity distribution for the three basic types of free-surface flows defined by the free-surface boundary conditions (30), i.e. for diffraction-radiation of water waves without forward speed, steady ship waves, and time-harmonic ship waves. The analytical

representations of waves given by (40) and (35) are remarkably simple. The Fourier-Kochin representations of waves obtained here from the velocity representation and in Noblesse and Yang [1] from the potential representation are in agreement. Practical applications of the Fourier-Kochin wave representation have previously been reported, for the case of steady ship waves, in Guillerm and Alessandrini [2] and Yang et al. [3,4].

The velocity representation, which has been used here within the Fourier-Kochin approach to obtain analytical representations of waves generated by a boundary velocity distribution, can also be used to obtain corresponding analytical representations of nearfield free-surface flows. Indeed, the velocity representation given in the present study, the Fourier-Kochin approach expounded in Noblesse and Yang [1] and used here, and the practical Fourier representation of super Green functions given in Noblesse and Chen [6] yield remarkably simple analytical representations of nearfield flows for diffraction-radiation of water waves without forward speed, steady ship waves, and time-harmonic ship waves. These analytical representations of nearfield flows, called Rankine and Fourier-Kochin nearfield flow representations, which extend the Fourier-Kochin wave representations given here, will be given elsewhere.

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## Appendix 1

The representation (10c) is now verified. The velocity component  $\vec{u}^F$  in (10a) can be expressed in the form (3a) where  $\vec{u}^\chi$  is defined by (3c) and (9) as  $\int_\Sigma d\mathcal{A} \vec{V}_1$  with

$$\vec{V}_1 = \begin{Bmatrix} (\phi G_{yy}^F + \phi G_{zz}^F) n^x - \phi G_{xy}^F n^y - \phi G_{xz}^F n^z \\ -\phi G_{yx}^F n^x + (\phi G_{zz}^F + \phi G_{xx}^F) n^y - \phi G_{yz}^F n^z \\ \phi G_{zx}^F n^x + \phi G_{zy}^F n^y - (\phi G_{xx}^F + \phi G_{yy}^F) n^z \end{Bmatrix}$$

Expression (10c) for  $\vec{u}^F = \vec{u}^\psi - \vec{u}^\chi$  holds if  $\vec{u}^\chi$  can be expressed as  $\int_\Sigma d\mathcal{A} \vec{V}_2$  with

$$\vec{V}_2 = \begin{Bmatrix} (\phi_x n^y - \phi_y n^x) G_y^F - (\phi_z n^x - \phi_x n^z) G_z^F \\ (\phi_y n^z - \phi_z n^y) G_z^F - (\phi_x n^y - \phi_y n^x) G_x^F \\ -(\phi_z n^x - \phi_x n^z) G_x^F + (\phi_y n^z - \phi_z n^y) G_y^F \end{Bmatrix}$$

i.e. if  $\int_\Sigma d\mathcal{A} (\vec{V}_1 - \vec{V}_2)$  is null. The  $x, y, z$  components of the vector  $\vec{V}_1 - \vec{V}_2$  are given by  $\vec{V}^x \cdot \vec{n}, \vec{V}^y \cdot \vec{n}, -\vec{V}^z \cdot \vec{n}$  where the vectors  $\vec{V}^x, \vec{V}^y, \vec{V}^z$  are defined by (6). The divergence theorem applied to the vector fields  $\vec{V}^x, \vec{V}^y, \vec{V}^z$  and the closed surface  $\Sigma$  completes the verification of (10c). This expression is at variance with (19c) in Noblesse et al. [8], which was obtained via an incorrect application of the divergence theorem to an open boundary surface.

## Appendix 2

A. Expressions (33a) and (35a) define the function  $S_\star^t$  in (36c) as

$$S_\star^t = \int_{\Sigma^{HW}} d\mathcal{A} [i\alpha (\vec{u} \times \vec{n})^x + i\beta (\vec{u} \times \vec{n})^y + k (\vec{u} \times \vec{n})^z] \mathcal{E}$$

where  $\mathcal{E} = \exp[kz + i(\alpha x + \beta y)]$ . We have

$$S_\star^t = \int_{\Sigma^{HW}} d\mathcal{A} \nabla \mathcal{E} \cdot (\vec{u} \times \vec{n}) = \int_{\Sigma^{HW}} d\mathcal{A} \vec{n} \cdot (\nabla \mathcal{E} \times \vec{u}) = \int_{\Sigma^{HW}} d\mathcal{A} \vec{n} \cdot (\nabla \times \mathcal{E} \vec{u})$$

since  $\nabla \times \vec{u} = 0$ . Stokes' theorem shows that  $S_\star^t$  is the function given by (35b).

B. Expressions (37) yield

$$\begin{cases} S^u + k S_f^u = -i\alpha S - i\beta S_\star^t / k + k S_f^u \\ S^v + k S_f^v = -i\beta S + i\alpha S_\star^t / k + k S_f^v \end{cases}$$

Thus, one needs to verify the relations

$$0 = \begin{Bmatrix} -i\beta S_*^t/k + k S_f^u + i\alpha S_*^\phi \\ i\alpha S_*^t/k + k S_f^v + i\beta S_*^\phi \end{Bmatrix} = \int_{\Gamma^{HW}} d\mathcal{L} \begin{Bmatrix} A^u \\ A^v \end{Bmatrix} \mathcal{E}_0$$

where  $\mathcal{E}_0 = \exp[i(\alpha x + \beta y)]$ . Expressions (33b) and (35b) yield

$$\int_{\Gamma^{HW}} d\mathcal{L} \begin{Bmatrix} A^u \\ A^v \end{Bmatrix} \mathcal{E}_0 = \frac{i}{k} \begin{Bmatrix} -\beta \\ \alpha \end{Bmatrix} \int_{\Gamma^{HW}} d\mathcal{L} [\vec{u} \cdot \vec{t} + i(\alpha t^x + \beta t^y) \phi] \mathcal{E}_0$$

We have

$$\int_{\Gamma^{HW}} d\mathcal{L} [\vec{u} \cdot \vec{t} + i(\alpha t^x + \beta t^y) \phi] \mathcal{E}_0 = \int_{\Gamma^{HW}} d\mathcal{L} \vec{t} \cdot \nabla (\phi \mathcal{E}_0) = 0 \quad (41)$$

which verifies the expressions for the spectrum functions related to  $u$  and  $v$  in (38a).

C. Expressions (37) yield

$$\begin{cases} S^\phi + F^2 S_F^\phi = S + S_*^\phi + F^2 S_F^\phi \\ S^u + F^2 S_F^u = -i\alpha S - i\beta S_*^t/k + F^2 S_F^u \\ S^v + F^2 S_F^v = -i\beta S + i\alpha S_*^t/k + F^2 S_F^v \end{cases}$$

Thus, one needs to verify the relations

$$0 = \begin{Bmatrix} F^2(S_F^\phi - S^F) + (1+D/k) S_*^\phi \\ F^2(S_F^u + i\alpha S^F) - i(1+D/k) S_*^t \beta/k \\ F^2(S_F^v + i\beta S^F) + i(1+D/k) S_*^t \alpha/k \end{Bmatrix} = F^2 \begin{Bmatrix} S_F^\phi - S^F + S_*^\phi \alpha^2/k \\ S_F^u + i\alpha S^F - i S_*^t \beta \alpha^2/k^2 \\ S_F^v + i\beta S^F + i S_*^t \alpha^3/k^2 \end{Bmatrix}$$

Here,  $D = F^2 \alpha^2 - k$ . These relations can be verified using (33c), (35b), and (41).

D. Expressions (38c) and (38b) show that one needs to verify the relations

$$\begin{Bmatrix} 2\tau S_\tau^\phi \\ f^2 S_f^u \\ f^2 S_f^v + 2\tau S_\tau^v \\ f^2 S_f^w + 2\tau S_\tau^w \end{Bmatrix} = \begin{Bmatrix} 1 \\ -i\alpha \\ -i\beta \\ k \end{Bmatrix} \left( \frac{f^2}{k} S_*^\phi - \frac{2\tau}{k} \frac{\beta}{k} S_*^t \right) + \frac{2\tau\alpha - f^2}{k} \vec{S}_*$$

Thus, one needs to verify the relations

$$0 = \begin{Bmatrix} k S_\tau^\phi - \alpha S_*^\phi + \beta S_*^t/k \\ k S_f^u + i\alpha S_*^\phi - i\beta S_*^t/k \\ k S_f^v + (2F/f) k S_\tau^v + i\beta S_*^\phi + i\alpha S_*^t/k - i(2F/f) k S_*^t \\ k S_f^w + (2F/f) k S_\tau^w - k S_*^\phi + (2F/f) \beta S_*^t \end{Bmatrix}$$

These relations can be verified using (33b), (33d), (35b), and (41).

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